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RESEARCH ARTICLE

A heuristic solution to Fermat's last theorem

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ABSTRACT

For centuries, notable mathematicians spent time to prove or refute Fermat's last theorem. As a result, many mathematical proofs have been generated for different values of *n*. The English mathematician Andrew Wiles, after many attempts, and with the help of his former student Richard Taylor, finally came up with a proof of Fermat's last theorem, which was published in 1995 in the prestigious journal *Annals of Mathematics*. Because centuries had passed without a proof of the theorem, many mathematicians suspected that Fermat had not developed a real proof. The demonstration presented by Andrew Wiles involved very complex mathematical operations that didn't exist in Fermat's time. Due to the high complexity of the demonstration of Fermat's last theorem, it is highly challenging to try to find solutions where it is mathematically valid. Therefore, and within this context, this paper aimed to present a heuristic solution of Fermat's last theorem using a specifically developed algebraic method. The purpose was not, under any circumstances, to propose a demonstration of the theorem, but to perform mathematical operations that aimed to facilitate its understanding, and consequently, its application. The outcomes obtained showed that it was possible to obtain valid solutions. This serves as a motivating element for teaching mathematics in undergraduate and even high school.

Keywords | Algebra. Trigonometry. Real Numbers. Math Analysis.

RESUMO / RESUMEN

Uma solução heurística para o último teorema de Fermat

Resumo | Durante séculos, notáveis matemáticos se debruçaram para provar ou refutar o último teorema de Fermat. Decorrente disso, foram geradas muitas provas matemáticas para diferentes valores de n. O matemático inglês Andrew Wiles, após várias tentativas, e com a ajuda de seu ex-aluno Richard Taylor, finalmente, apresentou uma prova do último teorema de Fermat, que foi publicado em 1995 no prestigiado periódico *Annals of Mathematics*. Pelo fato de ter se passado séculos sem uma prova do teorema, muitos matemáticos suspeitaram de que Fermat não desenvolveu uma prova real. A demonstração apresentada por Andrew Wiles envolveu operações matemáticas bastantes complexas que não existiam na época de Fermat. Devido à elevada complexidade da demonstração do último teorema de Fermat, é altamente desafiador tentar encontrar soluções onde ele seja matematicamente válido. Sendo assim, e dentro desse contexto, o presente trabalho teve como objetivo apresentar uma solução heurística do último teorema de Fermat usando um método algébrico especificamente desenvolvido. A finalidade não foi, em hipótese alguma, propor uma demonstração do teorema, mas, realizar operações matemáticas que visaram facilitar a sua compreensão, e consequentemente, sua aplicação. Os resultados obtidos mostraram que foi possível obter soluções válidas. Isso serve como elemento motivador para o ensino de matemática na graduação e até no ensino médio.

Palavras-chave | Álgebra. Trigonometria. Números Reais. Análise Matemática

Una solución heurística al último teorema de Fermat

Resumen | Durante siglos, matemáticos notables se han esforzado por probar o refutar el último teorema de Fermat. Como resultado, se generaron muchas demostraciones matemáticas para diferentes valores de n. El matemático inglés Andrew Wiles, tras varios intentos, y con la ayuda de su antiguo alumno Richard Taylor, finalmente presentó una demostración del último teorema de Fermat, que fue publicada en 1995 en la prestigiosa revista *Annals of Mathematics*. Debido a que habían pasado siglos sin una prueba del teorema, muchos matemáticos sospecharon que Fermat no había desarrollado una prueba real. La demostración presentada por Andrew Wiles involucraba operaciones matemáticas muy complejas que no existían en la época de Fermat. Debido a la alta complejidad de la demostración del último teorema de Fermat, es muy difícil tratar de encontrar soluciones donde sea matemáticamente válido. Por lo tanto, y dentro de este contexto, el presente trabajo tuvo como objetivo presentar una solución heurística del Último Teorema de Fermat utilizando un método algebraico específicamente desarrollado. El propósito no era, en ningún caso, proponer una demostración del teorema, sino realizar operaciones matemáticas que tuvieran como objetivo facilitar su comprensión y, en consecuencia, su aplicación. Los resultados obtenidos mostraron que era posible obtener soluciones válidas. Esto sirve como elemento motivador para la enseñanza de las matemáticas en pregrado e incluso bachillerato.

Palabras-clave | Álgebra. Trigonometría. Numeros Reales. Análisis Matemático.

Introduction

Fermat's last theorem is a theorem first proposed by Fermat in the form of a scribbled note in the margin of his copy of Diofanto's ancient Greek text *Arithmetica*. The scribbled note was discovered posthumously, and the original is lost. However, a copy has been preserved in a book published by Fermat's son (NAGELL, 1951). In the note, Fermat claimed to have discovered a proof that the Diophantine equation $x^n + y^n = z^n$ has no integer solutions for $n > 2$ with x, y, and z different from zero. The full text of Fermat's statement reads that: "It is impossible for a cube to be the sum of two cubes; a fourth power to be the sum of two fourth powers; or in general, for any number which is a power greater than two it is impossible for that number to be equal to the sum of two numbers of that same power. I have discovered a truly wonderful demonstration of this proposition which this margin is too narrow to contain. " (NAGELL, 1951).

As a result of Fermat's marginal note, the proposition that the Diophantine equation $x^n + y^n = z^n$; where: $x, y, z, and\ n$, are integers, has no non-zero solutions for $n > 2$, became known as Fermat's last theorem. It was called a "theorem" based on Fermat's statement, even though no other mathematician was able to prove it for hundreds of years (STEWART; TALL, 2000). One can observe that the restriction $n > 2$ is obviously necessary, since there are several elementary formulas for generating an infinite number of Pythagorean triples (x, y, z) triples that satisfy the equation for $n = 2$, $x^2 + y^2 = z^2$.

For centuries, many renowned mathematicians have tried to prove Fermat's last theorem but have been unsuccessful. However, in 1995, Andrew Wiles with the help of Richard Taylor, presented a definitive proof for the emblematic theorem. The proof of Fermat's last theorem marks the end of a mathematical era. Since virtually all the tools that were eventually used to solve the problem had not yet been invented in Fermat's time, it is interesting to speculate whether he possessed an elementary proof of the theorem. Judging by the tenacity with which the problem withstood attack for so long, Fermat's supposed proof seems to have been illusory. This conclusion is further supported by the fact that Fermat sought proofs for the cases $n = 4$, and $n = 5$ which would have been superfluous if he really had been in possession of a general proof (RIBENBOIM, 1999).

The demonstration presented by Andrew Wiles involved rather complex mathematical operations and well above the level of undergraduate mathematics education. However, this does not prevent Fermat's last theorem from being used in simpler applications within the context of basic mathematics. Thus, using elementary operations, the mathematical beauty of this theorem can be studied through the development of a heuristic method of interaction between algebra, arithmetic and geometry, including, in high school, as a stimulating resource within the teaching-learning process.

The heuristic method has been a very promising resource in solving mathematical problems at different learning levels. The core of mathematical problem solving is to explore and establish relationships between different branches of mathematical knowledge. Heuristic approaches can illustrate how these branches of knowledge are united by some basic and universal principles during the transfer of mathematical knowledge in the problem-solving process (HOON; KEE; SINGH, 2013). Thus, using the heuristic method to study Fermat's last theorem opens promising ways to understand its application in basic mathematics. It would be very important for the teaching-learning process the integration between the entities: heuristic method, Fermat's last theorem and problem solving.

Of all school subjects, mathematics introduces and develops the concept of "problem solving" as a fundamental component of school learning with a strong formative effect on students. In mathematics, problem solving represents the most effective concept for contextualizing and recontextualizing concepts, for transferring operational and basic mathematical knowledge to ensure sustainable and meaningful learning. The student's problem-solving behavior also involves, in addition to cognitive factors, factors aimed at the student's affectivity and experience (CĂPRIOARĂ, 2015).

Therefore, the objective of this study was to present a heuristic solution of Fermat's last theorem using basic operations of elementary mathematics. We emphasize that the purpose was not to make a demonstration of the theorem, but to create an approach as a learning resource within a theme that has been discussed for centuries.

Methodology

The heuristic method is a viable alternative to study Fermat's last theorem through simple mathematical operations of algebra, arithmetic, and geometry. Thus, it was necessary to start with an approach about the number sets until the final demonstration. The methodology was divided into three (3) important steps to facilitate the demonstration understanding.

Fermat's last theorem, also called Fermat's grand theorem, has the statement that there are no natural numbers x, y, and z such that $x^n + y^n = z^n$, For example, if $n = 3$, Fermat's last theorem states that there are no natural numbers x, y, and z *such that* $x^3 + y^3 = z^3$. That is, the sum of two cubes is not a cube.

Step 1. Writing x, y, and z as the ratio of two numbers.

Fermat's last theorem says that $x^n + y^n = z^n$ has no solution for $n \in \mathbb{N}$ and $n \geq 3$ with x, y *and* $z \in \mathbb{Z}$ or x, y and $z \in \mathbb{Q}$, because, $\mathbb{Z} \subset \mathbb{Q}$. In this paper we consider x, y and $z \in \mathbb{Q}$. Therefore, we can write x, y, and z as the ratio of two numbers:

$$x = \frac{a_1}{b_1} \in \mathbb{Q} \; with \; a_1, b_1 \in \mathbb{Q}$$

$$y = \frac{a_2}{b_2} \in \mathbb{Q} \; with \; a_2, b_2 \in \mathbb{Q}$$

$$z = \frac{a_3}{b_3} \in \mathbb{Q} \; with \; a_3, b_3 \in \mathbb{Q}$$

Step 2. Heuristic proposal.

Heuristically, in the present paper it was considered that: $x_1^n + y_1^n = z_1^n \rightarrow$ *with* $n \geq 3$ and $n \in \mathbb{N}$, x_1, y_1 and $z_1 \in \mathbb{Q}$.

$\mathbb{N} \rightarrow$ The natural numbers set.

$\mathbb{Z} \rightarrow$ The integers numbers set.

$\mathbb{Q} \rightarrow$ The rational numbers set.

$\mathbb{Z} \subset \mathbb{Q} \rightarrow$ Every integer is rational.

$\mathbb{Q} \not\subset \mathbb{Z}$

$(z_1^n)^2 = (z_1^2)^n \rightarrow$ One property of power operations.

Step 3. Study Hypothesis.

To proceed with the heuristic proposal and analyzing its feasibility, the following systems were considered:

(I) $x_1^n + y_1^n = z_1^n$

(II) $(x - x_1^n) \cdot (x - y_1^n) = 0$

With $x_1, y_1,$ and $z_1 \in \mathbb{Q}$; $n \geq 3$ and $n \in \mathbb{N}$.

Results and discussion

Considering the 1st Step of the methodology, for $x^n + y^n = z^n$, we have:

$\left(\frac{a_1}{b_1}\right)^n + \left(\frac{a_2}{b_2}\right)^n = \left(\frac{a_3}{b_3}\right)^n$ has no solution for n $\geq$ 3,for, $\left(\frac{a_1}{b_1}\right)^n + \left(\frac{a_2}{b_2}\right)^n = \frac{(a_3)^n}{(b_3)^n}$ $\rightarrow \frac{(a_1b_2b_3)^n+(a_2b_1b_3)^n}{(b_1b_2b_3)^n} = \frac{(a_3b_1b_2)^n}{(b_1b_2b_3)^n} \therefore (a_1b_2b_3)^n + (a_2b_1b_3)^n = (a_3b_1b_2)^n$, with $(a_1b_2b_3);\ (a_2b_1b_3)\ and\ (a_3b_1b_2) \in \mathbb{Q}$. From mathematical analysis we know that: Let $v_1 \in \mathbb{R}$, $v_1 \geq 0$, $v_2 \in \mathbb{R}$, $v_2 \geq 0$ then: $0 \leq \frac{(v_1^n - v_2^n)^2}{(v_1^n + v_2^n)^2} \leq 1\ with\ n\ \in \mathbb{N}$;

$n \geq 3\ \ and\ (v_1^n - v_2^n)^2 \leq (v_1^n + v_2^n)^2$. Considering simultaneously the steps 2 and 3, we will have:

(I) $\frac{x_1^n + y_1^n}{z_1^n} = 1$

(II) $x^2 - (x_1^n + y_1^n)x + (x_1y_1)^n = 0$

$$\frac{x^2}{z_1^n} - \left(\frac{x_1^n + y_1^n}{z_1^n}\right)x + \left(\frac{x_1y_1}{z_1}\right)^n = 0$$

$$\frac{x^2}{z_1^n} - 1x + \left(\frac{x_1\,y_1}{z_1}\right)^n = 0$$

But, $\Delta = (b)^2 - 4\,(a)(c)$, then:

$\Delta = (-1)^2 - 4\left(\frac{1}{z_1^n}\right)\left(\frac{x_1\,y_1}{z_1}\right)^n = 1 - 4\left(\frac{1}{z_1^n}\right)\left(\frac{x_1\,y_1}{z_1}\right)^n$

$\Delta = 1 - 4\left(\frac{x_1\,y_1}{z_1^2}\right)^n$

Replacing $z_1 = (x_1^n + y_1^n)^{\frac{1}{n}}$ in Δ, we have:

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = 1 - 4\left(\frac{x_1\,y_1}{(x_1^n + y_1^n)^{\frac{2}{n}}}\right)^n$

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = 1 - \frac{4(x_1\,y_1)^n}{(x_1^n + y_1^n)^2}$

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = \frac{(x_1^n + y_1^n)^2 - 4\,(x_1y_1)^n}{(x_1^n + y_1^n)^2}$

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = \frac{x_1^{2n} + 2(x_1y_1)^n + y_1^{2n} - 4(x_1y_1)^n}{(x_1^n + y_1^n)^2}$

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = \frac{x_1^{2n} - 2\,(x_1y_1)^n + y_1^{2n}}{(x_1^n + y_1^n)^2}$

$\Delta = 1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2}$. Here, we have a perfect square that can be greater than or equal to zero. So, this implies that:

$1 - 4\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n \geq 0$, ie, $2\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n \leq ½$.

As $0 \leq \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2} \leq 1$ e $2\left\{\frac{x_1\,y_1}{z_1^2}\right\}^n \leq ½$ (com $x_1 \neq y_1,\ x_1 = y_1$, $x_1 \geq 0$, $y_1 \geq 0$ e $z_1 > 0$), and making the intersection between these two intervals, we come to the conclusion that the validity interval for Δ it is $0 \leq 2\left(\frac{x_1\,y_1}{z_1^2}\right)^n \leq ½$.

Let us note that:

(**A**) $0 \leq \frac{(x_1^n-y_1^n)^2}{(x_1^n+y_1^n)^2} \leq 1$ for $x_1 \neq y_1$, $x_1 = y_1$, $x_1 \geq 0, y_1 \geq 0$ 0, with $\frac{(x_1^n-y_1^n)^2}{(x_1^n+y_1^n)^2} \in \mathbb{Q}$ and $(x_1^n - y_1^n)^2 \leq (x_1^n + y_1^n)^2$.

(**B**) $\frac{(x_1^n-y_1^n)^2}{(x_1^n+y_1^n)^2}$ is a perfect square.

It is known that: $sen^2\theta + cos^2\theta = 1, \forall\, \theta \in \,]\,0, 2\pi\,[$.

Replacing $sen^2\theta + cos^2\theta = 1$ in Δ, we have:

$\Delta = sen^2\theta - 4\left(\frac{x_1\, y_1}{z_1^2}\right)^n + cos^2\theta = \left(\frac{x_1^n-y_1^n}{x_1^n+y_1^n}\right)^2$

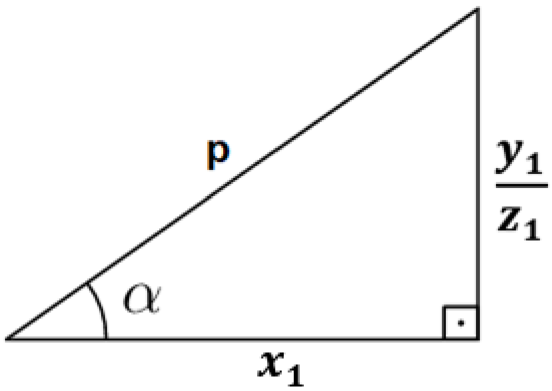
What is the conditition for $sen^2\theta - 4\left\{\frac{x_1\, y_1}{z_1^2}\right\}^n + cos^2\theta \in \mathbb{Q}$, be a perfect square? One of the conditions is:

$-4\left(\frac{x_1\, y_1}{z_1^2}\right)^n = -2\, cos\theta sen\theta$ or $2\left\{\frac{x_1\, y_1}{z_1^2}\right\}^n = cos\theta sen\theta$.

With: $0 \leq 2\left(\frac{x_1\, y_1}{z_1^2}\right)^n \leq 1/2$, and $sen\theta \in \mathbb{Q}$, and $cos\theta \in \mathbb{Q}$, we will have:

$sen^2\theta - 4\left\{\frac{x_1\, y_1}{z_1^2}\right\}^n + cos^2\theta = sen^2\theta - 2\, cos\theta sen\theta + cos^2\theta = (sen\theta - cos\theta)^2$ with Δ being a perfect square. Taking the equality analysis:

$2\left(\frac{x_1\, y_1}{z_1^2}\right)^n = (cos\theta sen\theta)$, and as declared before, $\forall\, \theta \in \,]\,0, 2\pi\,[$, then, it is possible to choose a value for θ. Therefore, consider the relationship in the right triangle below.



Let us note that:

$(p)^2 = (x_1)^2 + \left(\frac{y_1}{z_1}\right)^2$

$p^2 \in \mathbb{Q}$.

$sen(\alpha) = \frac{\left(\frac{y_1}{z_1}\right)}{p} = \frac{y_1}{pz_1} \Rightarrow \frac{y_1}{z_1} = psen(\alpha) \Rightarrow y_1 = pz_1 sen(\alpha)$

$cos(\alpha) = \frac{x_1}{p} \Rightarrow x_1 = pcos(\alpha)$

The inequality considered here will be $0 < 2\left(\frac{x_1\, y_1}{z_1^2}\right)^n \leq ½$ because they are the sides of a triangle.

Choosing $\theta = \alpha$ with $0 < \theta \leq \frac{\pi}{4}$, then:

$\omega = 2\,(\frac{x_1\,y_1}{z_1^2})^n = 2\,(\frac{y_1}{z_1}\,.\,x_1.\frac{1}{z_1})^n = \frac{2.(1)^n}{z_1^n}.(\frac{y_1}{z_1}.\,x_1)^n = \cos(\theta)sen(\theta)$

Due to inequality $0 < 2\,(\frac{x_1\,y_1}{z_1^2})^n \leq ½$, so the angle θ it's at most $\frac{\pi}{4}$, because for this value $\cos(\theta)sen(\theta) = \frac{1}{2}$. We consider zero less than $2\,(\frac{x_1\,y_1}{z_1^2})^n$, because it's the sides of a triangle.

Replacing $y_1 = pz_1sen(\theta)$ and $x_1 = pcos(\theta)$ into ω, one has:

$\omega = \frac{2}{z_1^n}\,\{\frac{p\,z_1 sen(\theta)}{z_1}.\,pcos(\theta)\}^n = \cos(\theta)sen(\theta)$

$\omega = \frac{2}{z_1^n}\,\{p^2 sen(\theta).\cos(\theta)\}^n = \cos(\theta)sen(\theta)$

$\omega = \frac{2\{p^2\}^n}{z_1^n}.\,\{sen(\theta)\cos(\theta)\}^n = \cos(\theta)sen(\theta)$

$2\{p^2\}^n.\,\{\cos(\theta)sen(\theta)\}^n = \{\cos(\theta)sen(\theta)\}.\,z_1^n$

$\frac{z_1^n}{(p^2)^n} = \frac{2\{sen(\theta)\cos(\theta)\}^n}{\{sen(\theta)\cos(\theta)\}}$

$\frac{z_1^n}{(p^2)^n} = 2\{sen(\theta)\cos(\theta)\}^{n-1}$

Replacing $sen(\theta) = \frac{y_1}{p\,z_1}$, and $\cos(\theta) = \frac{x_1}{p}$, we have:

$\frac{z_1^n}{(p^2)^n} = 2\,(\frac{y_1}{pz_1}\,.\frac{x_1}{p})^{n-1}$

$\frac{z_1^n}{(p^2)^n} = 2\,(\frac{x_1y_1}{p^2\,z_1})^{n-1}$

Considering $x_1 = 2\,v_1$ with $v_1 \in \mathbb{Q}$ and replacing $x_1^n + y_1^n = z_1^n$, then, we have:

$\frac{x_1^n+y_1^n}{(p^2)^n} = 2(\frac{2(v_1y_1)}{p^2\,z_1})^{n-1}$

$\frac{(x_1)^n}{(p^2)^n} + \frac{(y_1)^n}{(p^2)^n} = 2.\,2^{n-1}\,(\frac{v_1y_1}{p^2\,z_1})^{n-1}$ ou

$\left(\frac{x_1}{p^2}\right)^n + \left(\frac{y_1}{p^2}\right)^n = (2)^n\,(\frac{v_1y_1}{p^2\,z_1})^{n-1}$

$\left(\frac{x_1}{2p^2}\right)^n + \left(\frac{y_1}{2p^2}\right)^n = (\frac{v_1y_1}{p^2\,z_1})^{n-1}$ ou $(x_1)^n + (y_1)^n = (2p^2)^n\,(\frac{v_1y_1}{p^2\,z_1})^{n-1}$

As $(2p^2)^n\,(\frac{v_1y_1}{p^2\,z_1})^{n-1} \neq (z_1)^n$, so stop $n \geq 3$, implies that $(x_1)^n + (y_1)^n \neq (z_1)^n$, that is, equation I of the system cannot take the form $(x_1)^n + (y_1)^n = (z_1)^n$, according to the conditions established in the work and the interval for which the system is valid.

Whereas that:

$\frac{x_1}{2p^2} = q_1$, $\frac{y_1}{2p^2} = q_2$, and $\frac{v_1y_1}{p^2\,z_1} = q_3$

Then, $(q_1)^n + (q_2)^n = (q_3)^{n-1}, with\ n\ \geq 3$ and $q_1, q_2\ and\ q_3 \in \mathbb{Q}$.

Based on what has been demonstrated so far, the following observations can be made:

(I) If $(q_1)^n + (q_2)^n = (q_3)^{n-1}$, then, $(q_1)^n + (q_2)^n$ cannot be written in the form $(q_1)^n + (q_2)^n = (q_3)^n$ with q_1, q_2, *and* $q_3 \in \mathbb{Q}$; with n≥ 3, and n $\in \mathbb{N}$.

(II) If $(q_1)^n + (q_2)^n$ cannot be written in the form $(q_1)^n + (q_2)^n = (q_3)^n$, then it is not valid that $x_1^n + y_1^n = z_1^n$ (with x_1, y_1, *and* $z_1 \in \mathbb{Q}$, with n≥ 3 and n $\in \mathbb{N}$) according to the parameters established earlier in this paper.

(III) If there are q_1, q_2, *and* q_3 with q_1, q_2, *and* $q_3 \in \mathbb{Q}$ such that $(q_1)^n + (q_2)^n = (q_3)^{n-1}$, then:

$(q_1)^n + (q_2)^n = (q_3)^{n-1} \Longrightarrow x_1^n + y_1^n \neq z_1^n$, with n $\geq$ 3 and n $\in \mathbb{N}$.

Thus, for n = 3 $\Longrightarrow (q_1)^3 + (q_2)^3 = (q_3)^2 \Longrightarrow (x_1)^3 + (y_1)^3 \neq (z_1)^3$.

For n = 4 $\Longrightarrow (q_1)^4 + (q_2)^4 = (q_3)^3 \Longrightarrow (x_1)^4 + (y_1)^4 \neq (z_1)^4$.

For n = 5 $\Longrightarrow (q_1)^5 + (q_2)^5 = (q_3)^4 \Longrightarrow (x_1)^5 + (y_1)^5 \neq (z_1)^5$.

....

For n = n $\Longrightarrow (q_1)^n + (q_2)^n = (q_3)^{n-1} \Longrightarrow (x_1)^n + (y_1)^n \neq (z_1)^n$.

The system for "n" greater than or equal to 3, the equation I cannot take the form $(x_1)^n + (y_1)^n = (z_1)^n$, that is, it can take any other form except the form $(x_1)^n + (y_1)^n = (z_1)^n$.

To provide further clarification, consider the following numerical example:

$$\left(\frac{1}{25}\right)^3 + \left(\frac{2}{25}\right)^3 = \left(\frac{15}{625}\right)^2$$

$$\left(\frac{1}{5^2}\right)^3 + \left(\frac{2}{5^2}\right)^3 = \left(\frac{3.5}{5^4}\right)^2$$

$$\frac{1}{5^6} + \frac{8}{5^6} = \frac{9}{5^6} \quad \text{ou} \quad \frac{1^3}{5^6} + \frac{2^3}{5^6} = \frac{3^2}{5^6} \Longrightarrow (1)^3 + (2)^3 = (3)^2$$

$$\Longrightarrow (1)^3 + (2)^3 = (5^2)^3.\left(\frac{3.5}{5^4}\right)^2$$

In the context analyzed in the present work to obtain a heuristic demonstration of Fermat's last theorem, it was possible to prove the hypotheses considered, and test them according to the procedures below.

I → We have seen that $\left(\frac{x_1}{2p^2}\right)^n + \left(\frac{y_1}{2p^2}\right)^n = \left(\frac{v_1 y_1}{p^2\, z_1}\right)^{n-1}$

Assuming:

$x_1 = 2p^2 a_1$ with $a_1 \in \mathbb{Q}$ and $p^2 \in \mathbb{Q} \Longrightarrow x_1 \in \mathbb{Q}$.

$y_1 = 2p^2 b_1$ with $b_1 \in \mathbb{Q}$ and $p^2 \in \mathbb{Q} \Longrightarrow y_1 \in \mathbb{Q}$.

$v_1 = \frac{c_1 p^2 z_1}{y_1}$ with $c_1, y_1, z_1 \in \mathbb{Q}$ and $p^2 \in \mathbb{Q} \Longrightarrow v_1 \in \mathbb{Q}$.

Then replacing in I:

$$\left(\frac{2p^2a_1}{2p^2}\right)^n + \left(\frac{2p^2b_1}{2p^2}\right)^n = \left(\frac{\frac{c_1}{y_1}.p^2z_1.y_1}{p^2z_1}\right)^{n-1}$$

$(a_1)^n + (b_1)^n = (c_1)^{n-1}$ when considering the parameters established in the paper, with a_1, b_1, and $c_1 \in \mathbb{Q}$, and $n \geq 3$ and $n \in \mathbb{N}$.

Example:

$$\frac{1^3}{5^6} + \frac{2^3}{5^6} = \frac{3^2}{5^6} \implies (1)^3 + (2)^3 = (3)^2$$

Interesting that for n equal two the system:

(I) $x_1^2 + y_1^2 = z_1^2$

(II) $(x - x_1^2)(x - y_1^2) = 0 \implies x^2 - (x_1^2 + y_1^2)x + (x_1y_1)^2 = 0 \implies x^2 - (z_1^2)x + (x_1y_1)^2 = 0,$

will have solutions x, x_1, y_1, z_1 that satisfy the two system equations with x_1, y_1, z_1 belonging to $\mathbb{Q}$.

Example:

The pythagorean trio $x_1 = 3$, $y_1 = 4$, $z_1 = 5$ and $x = 9$ or $x_1 = 3$, $y_1 = 4$, $z_1 = 5$, and $x = 16$ are values that satisfy both equations of the system.

Substituting into 1 the expression $-\,4\,\{\frac{x_1\,y_1}{z_1^2}\}^n = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2}$, we will have:

$$1 - 4\,\{\frac{3.4}{5^2}\}^2 = \frac{(3^2-4^2)^2}{(3^2+4^2)^2}$$

1-144/625=49/625

49/625=49/625

For n equals one, $x_1 = 2$, $y_1 = 7$, $z_1 = 9$, and $x = 2$ or $x_1 = 2$, $y_1 = 7$, $z_1 = 9$, and $x = 7$ are values that satisfy the two system equations below:

(I) $x_1 + y_1 = z_1$

(II) $(x - x_1)(x - y_1) = 0$

Substituting into 1 the expression $-\,4\,\{\frac{x_1\,y_1}{z_1^2}\}^n = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2}$, we will have:

$$1 - 4\{\frac{2.7}{9}\} = \frac{(2-7)^2}{(2+7)^2}$$

1-56/81=25/81

25/81=25/81

However, if n is greater than or equal three, then it will not be possible to exist x_1, y_1 and z_1 belonging to $\mathbb{Q}$ that satisfy both equations of the system, i.e., the system for $n \geq 3$ must be as follows:

(I) $(x_1)^n + (y_1)^n = (2p^2)^n \cdot (\frac{v_1y_1}{p^2\,z_1})^{n-1}$

(II) $(x - x_1^n)(x - y_1^n) = 0$

Let us note that, $x_1 = 1$, $y_1 = 2$, $z_1 = 3$, and $x = 8$, or $x_1 = 1$, $y_1 = 2$, $z_1 = 3$, and $x = 1$ are values that satisfy both equations of the system for n = 3.

If we replace in 1 the expression $- 4 \{\frac{x_1 y_1}{z_1^2}\}^n = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2}$, would not work, because for $n \geq 3$, changed the equation I of the system and changed the form in relation to the system proposed at the beginning of the work.

Let us note that, $\text{sen}^2\theta - 4 \{\frac{x_1 y_1}{z_1^2}\}^n + \cos^2\theta = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2}$ implies that:

$\text{sen}^2\theta + \cos^2\theta = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2} + 4 \{\frac{x_1 y_1}{z_1^2}\}^n$

$\text{sen}^2\theta + \cos^2\theta = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2} + 4\frac{(x_1 y_1)^n}{(z_1^n)^2} = \frac{(x_1^n - y_1^n)^2}{(x_1^n + y_1^n)^2} + 4\frac{(x_1 y_1)^n}{(x_1^n + y_1^n)^2} = \frac{(x_1^n + y_1^n)^2}{(x_1^n + y_1^n)^2} = 1$. This shows that the heuristic hypotheses are true for the presented demonstration.

Final considerations

Although the demonstration of Fermat's last theorem involves rather complex mathematical operations, the present work presented a feasible heuristic solution for simple application.

Obtaining $(q_1)^n + (q_2)^n = (q_3)^{n-1}$ through basic mathematical operations showed the possibility of studying Fermat's last theorem in a more simplified, but not simplistic, way.

The mathematical heuristic method proved to be an efficient tool for finding a simple solution to Fermat's last theorem.

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